Math 8

Homework Set #7

Convergence Test Practice

## Practice Problems

Use any of the convergence tests we have learned in class to determine if the following series converge or diverge.

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{3^n}$

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} + \left(\frac{5}{4}\right)^n$

3) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$

4) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{1+n^3}$

5) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$

6) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3+5}{7n^2-11n^3}$

7) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{n\pi}{2(n+1)}\right)$

8) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

9) $\displaystyle\sum_{n=1}^{\infty} \frac{3+\cos n}{e^n}$

10) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

11) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n}$

12) $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{(1+n)^3}$

13) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 3n}{1+2^n}$

14) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)^n$

15) $\displaystyle\sum_{n=1}^{\infty} n\left(e^{1/n}-1\right)$

16) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+2}$

17) $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}-1$

1) $\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{3^n}$

Note $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{3^n}$ are convergent geometric series.

Then $\sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{3^n}$ converges.

2) $\sum_{n=1}^{\infty} \frac{1}{n^3} + \left(\frac{5}{4}\right)^n$ diverges by the divergence test as $\lim_{n\to\infty} \frac{1}{n^3} + \left(\frac{5}{4}\right)^n$ diverges to $\infty$

3) $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$ converges by the comparison test

as $0 \le \frac{1}{2^n+3^n} \le \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series

4) $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges by the comparsison test

as $\frac{n^2}{n^3+1} > \frac{n^2}{n^3+n^3} = \frac{1}{2n}$ and $\sum_{n=1}^{\infty} \frac{1}{2n}$ is harmonic, hence diverges

5) $\sum_{n=1}^{\infty} \frac{n^n}{3^n\, n!}$ converges by the ratio test:

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1} / (n+1)!\, 3^{n+1}}{n^n / n!\, 3^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \frac{n!\, 3^n}{(n+1)!\, 3^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{(n+1)3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n}{3n^n} \right| = \lim_{n\to\infty} \frac{1}{3} \left( \frac{n+1}{n} \right)^n = \frac{1}{3} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \frac{e}{n} < 1$$

6) $\sum_{n=1}^{\infty} \frac{4n^3+5}{7n^2-11n^3}$ diverges by the divergence test

$$\lim_{n\to\infty} \frac{4n^3+5}{7n^2-11n^3} = -\frac{4}{11} \neq 0$$

why?

7) $\sum_{n=1}^{\infty} (-1)^n \cos(n\pi/(2n+1))$ converges by the alternating series test:

$\lim_{n\to\infty} \cos\left(\frac{n\pi}{(2n+1)}\right) = \cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{(n+1)\pi}{2(n+1)+1}\right) \le \cos\left(\frac{n\pi}{2n+1}\right)$

as $\frac{d}{dx}\cos(x) = -\sin(x) < 0$ for $0 < x < \frac{\pi}{2}$

and $\frac{n\pi}{2n+1} < \frac{(n+1)\pi}{2(n+1)+1} < \frac{\pi}{2}$

Check this!

8) $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ converges by the root test:

$$\lim_{n\to\infty} \left| \frac{(2n+1)^n}{n^{2n}} \right|^{1/n} = \lim_{n\to\infty} \frac{2n+1}{n^2} = 0 < 1$$

9) $\sum_{n=1}^{\infty} \frac{3+\cos(n)}{e^n}$ converges by the comparison test:

$0 < \frac{3+\cos n}{e^n} < \frac{4}{e^n}$ with $4\sum_{n=1}^{\infty} e^n$ a convergent geometric series

10) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ diverges by the integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \int_2^{\infty} \frac{1}{\sqrt{u}}\,du = \left[2\sqrt{u}\right]_2^{\infty} = \left[2\sqrt{\ln(x)}\right]_2^{\infty} = \lim_{x\to\infty} 2\sqrt{\ln(x)} - 2\sqrt{\ln(2)}$$

$u=\ln(x)$, $du = \frac{1}{x}dx$

diverges to $\infty$

11) $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n} = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{e^n}$ converges by the root test
(or ratio test or alternating series test)

$$\lim_{n\to\infty} \left| \frac{(-1)^n n^2}{e^n} \right|^{1/n} = \lim_{n\to\infty} \frac{n^{2/n}}{e} = 1/e < 1$$

12) $\sum_{n=1}^{\infty} \frac{n\ln(n)}{(1+n)^3}$ converges by the comparison test:

$\ln(n) < \sqrt{n}$ so $0 < \frac{n\ln(n)}{(1+n)^3} < \frac{n\sqrt{n}}{n^3} = \frac{1}{n^{3/2}}$, a convergent p-series

13) $\sum_{n=1}^{\infty} \frac{\sin(3n)}{1+n^3}$ converges by the comparison test:

$|\sin(3n)| \leq 1$, so $\left| \frac{\sin(3n)}{1+n^3} \right| \leq \frac{1}{1+n^3} < \frac{1}{n^3}$, a convergent p-series

Then the series converges <u>absolutely</u>, hence converges.

14) $\sum_{n=1}^{\infty} (\sqrt{2}^{1/n}-1)^n$ converges by the root test:

$$\lim_{n\to\infty} \left|(\sqrt{2}^{1/n}-1)^n\right|^{1/n} = \lim_{n\to\infty} \sqrt{2}^{1/n}-1 = 1-1=0<1$$

15) $\sum_{n=1}^{\infty} n(e^{1/n}-1)$ diverges by the divergence test:

$$\lim_{n\to\infty} n(e^{1/n}-1) = \lim_{n\to\infty} n\left(\frac{1}{n}+\frac{(1/n)^2}{2!}+\frac{(1/n)^3}{3!}+\dots\right) = \lim_{n\to\infty}\left(1+\frac{(1/n)}{2!}+\frac{(1/n)^2}{3!}+\dots\right)=1$$

16) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+2}$ converges by the alternating series test:

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+2}=0 \text{ and } \frac{\sqrt{n+1}}{n+3} \leq \frac{\sqrt{n}}{n+2} \text{ as } \frac{d}{dx}\sqrt{x}\,(x+2)^{-1} = \frac{1}{2}\frac{1}{\sqrt{x}\,(x+2)} - \frac{\sqrt{x}}{(x+2)^2}$$

why?

$$= \frac{1}{2}\frac{\sqrt{x}}{x(x+2)} - \frac{\sqrt{x}}{x^2+4x+4}$$

$$= \sqrt{x}\left(\frac{1}{2x^2+4x} - \frac{1}{x^2+4x+4}\right) < 0$$

$2x^2+4x > x^2+4x+4$ for $x>1$

17) $\sum_{n=1}^{\infty} \sqrt[n]{2}-1 = \sum_{n=1}^{\infty} e^{\ln(\sqrt[n]{2})}-1 = \sum_{n=1}^{\infty} e^{\frac{1}{n}\ln(\sqrt{2})}-1$ diverges by the comparison test

$$e^{1/n\ln(2)}-1 = \frac{1}{n}\ln(2) + \frac{\left(\frac{1}{n}\ln(2)\right)^2}{2!} + \frac{\left(\frac{1}{n}\ln(2)\right)^3}{3!}+\dots > \ln(2)\frac{1}{n} \text{ with } \sum_{n=1}^{\infty}\frac{\ln(2)}{n} \text{ divergent}$$

(harmonic)